**2 A Closed Formula for the Symmetric Group *S\_n***

Before presenting the formula for generating all permutations of the symmetric group *S\_n*, we will describe what is a **permutation**, then show some initial cases of generating permutations, and then finally present the formula.

**Definition 2.1.** *Permutations are functions over a finite product of a totally ordered set to itself, which rearrange the terms of the product set. That is, if A is a set, and n a natural number, then τ = [j\_1j\_2 ... j\_n] denotes a* ***permutation, τ*** *: A x A x A ... A --> A x A ... x A (n times), which rearranges the input (a\_1, a\_2, ..., a\_n) with respect to τ, [j\_1j\_2 ... j\_n], where j\_1, j\_2, ... , j\_n are in {1, 2, ... , n} and, j\_i <> j\_k if and only if, i <> k, i, k in {1, 2, ... , n} so that 1st location of the n-tuple (a\_1, a\_2, ... , a\_n) takes the value a\_{j\_1} and the j\_1st location takes the value a\_1, the 2nd the value of a\_{j\_2} and the j\_2nd location the value a\_2, and so on.*

**Definition 2.2.** *Transpositions are functions as permutations, only that a* ***transposition σ****, is a [ij], where the i-th location of the tuple (a\_1, a\_2, ..., a\_n) gets the a\_j and the j-th location the a\_i. That is an interchange of terms i, j.*

From the above two definitions, one can say that a permutation is composed of at least one transposition. The set of permutations is a finite group, called *symmetric group S\_n of order n!*, with composition law the trivial function composition, and special ("unit") element the permutation τ = [123...n].

Now, let's try a method of generating all permutations on initial cases. Consider for the beginning we get [1234].



From Figure 2.1., one can see that the following formula can be derived which generates the group *S\_n*. Let first denote the equalities (1):

A\_1 = [12]

A\_2 = [23]

A\_j = [j(j+1)]

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A\_(n-1) = [(n-1)n]

**Theorem 2.1.** *The formula:*

*(A\_{j\_1} A\_{j\_2} ... A\_{j\_{k\_1}})^N\_1 (A\_{i\_1} A\_{i\_2} ... A\_{i\_{k\_2}})^N\_2 ... (A\_{s\_1} A\_{s\_2} ... A\_{s\_{k\_m}})^N\_m, (2),*

*where j\_1, j\_2, ...,j\_{k\_1}, i\_1, i\_2, ..., i\_{k\_2},..., s\_1, s\_2, ..., s\_{k\_m} are in {1, 2, ... (n-1)}, and 0 <= k\_j <= n - 1, 1 <= j <= m, and N\_j, 1 <= j <= m, can be any natural number, generates all possible permutations of the group S\_n.*

*Proof:* It will be proved by applying induction on the number of transpositions a permutation can be broken down. Initially assume that a permutation τ is composed from a single transposition [ij] (j = i + m, m > 0). Then the effect of this transposition is equivalent to applying the composition of transpositions, [(j - 1)j] ... [(i + 1)(i + 2)] [i(i + 1)] followed by the composition of transpositions [i(i + 1)] ... [(j - 2)(j - 1)]. (Recall that transpositions are function, and composition of transpositions are function composition applied to an n-tuple.) By the equalities (1), we have (A\_i A\_(i + 1) ... A\_(j - 2) A\_(j - 1) A\_j ... A\_(i + 1) A\_i), which is a straightforward application of (2) with N\_m = 1, with m = 1.

Assume now that any permutation τ composed from k transpositions is described by (2) that is, τ = (A\_{j\_1} A\_{j\_2} ... A\_{j\_{k\_1}})^N\_1 (A\_{i\_1} A\_{i\_2} ... A\_{i\_{k\_2}})^N\_2 ... (A\_{s\_1} A\_{s\_2} ... A\_{s\_{k\_m}})^N\_m. Of course, **Σ**N\_i \* k\_i = k, where 1 <= i <= m. Subsequently, consider the application of one extra transposition σ = [ij] on τ. Then στ = [ij]τ. The transposition σ, as we saw in the initial case of the induction, breaks down to a form as (2). Therefore, στ has the form of (2). QED

**Remark 2.1.** We can further see that **Σ**N\_i, 1 <= i <= m, is bounded below. To see that, one can write (2) as,

(A^(0,1)\_1 A^(0,1)\_2 ... A^(0,1)\_{(n - 1)})^N\_1 (A^(0,1)\_1 A^(0,1)\_2 ... A^(0,1)\_{(n - 1)})^N\_2 ... (A^(0,1)\_1 A^(0,1)\_2 ... A^(0,1)\_{(n - 1)})^N\_m,

where A^(0,1) means that either the transpositions as in (1) appears, or not. Therefore, we have at most M = 2^{(n - 1) \* **Σ**N\_i} (1 <= i <= m) permutations generated by the formula (2). Therefore M >= n! => (n - 1) \* **Σ**N\_i >= **Σ**logj, (1 <= j <= n) => **Σ**N\_i = o(**Σ**logj/(n - 1)) (1 <= j <= n) (3). That is, it suffices to apply (2) at least [**Σ**logj/(n - 1)] + 1 on **Σ**N\_i, to generate *S\_n*. ([x] gives the integer part of the number x.)

A piece of algorithmic code that generates all permutations applying (2) that will be used to the series of quantum algorithms in the coming sections, is as below:

S\_n = {}

M = [**Σ**logj/(n - 1)] + 1

**for** N\_1, N\_2, ..., N\_M = 1 **to** **Σ**N\_i ( 1<= i <= M) = 1 **to** M **do (\*)**

**for** b\_{(1,1)}, b\_{(1, 2)}, ..., b\_{(1,(n - 1))} = 0, 1 **and** at least 1 of them <> 0 **do**

**for** b\_{(2,1)}, b\_{(2, 2)}, ..., b\_{(2,(n - 1))} = 0, 1 **and** at least 1 of them <> 0 **do**

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**for** b\_{((n - 1),1)}, b\_{((n - 1), 2)}, ..., b\_{(M,(n - 1)))} = 0, 1 **and** at least 1 of them <> 0 **do**

S\_n= S\_n **U Π**\_m (A^{b\_{(m,1)}}\_1 A^{b\_{(m, 2)}}\_2 ... A\_^{b\_{((m,(n - 1))}}\_{(n - 1)})^N\_m (1 <= m <= M)

**end**

**end**

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.

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**end**

**end**

**end**

Here there is a useful point that I'd like to explain. How many iterations are assumed in command **(\*)**? We will sketch to start, the cases k\_1 + k\_2 + k\_3 = 3. We can see that the possible cases are produced by the below recursive SUM of the tree in Figure 2.2(a). Then we generalize to the case k\_1 + k\_2 + ...+ k\_m = m in Figure 2.2(b), which are special cases of the recursive tree of Figure 2.2(c), where k\_1 + k\_2 + ...+ k\_m = ν, ν <> 0. Finding the number of possible sums is equivalent to constructing the recursive tree in Figure 2.2, and following all paths from the root to the leaves.







It is straightforward that all paths from the root to the leaves are O(m^m). Further, we can take a lower bound estimate on the paths towards the leaves by noticing that the binary tree of height (m - 1) is an isomorphic sub-graph to the recursive tree of Figure 2.2(b). Therefore, the possible sums are o(2^m).